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A RESUME OF STOCHASTIC, TIME-VARYING, LINEAR SYSTEM THEORY WITH APPLICATION TO ACTIVE-SONAR SIGNAL-PROCESSING PROBLEMS

by

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Director

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A RESUME OF STOCHASTIC, TIME-VARYING, LINEAR SYSTEM THEORY WITH APPLICATION TO ACTIVE-SONAR SIGNAL-PROCESSING PROBLEMS

by

Lewis Meier

ABSTRACT

This report summarizes linear time-varying systems theory and its application to active sonar signal processing when certain of the parameters are random variables. To keep the analysis within reasonable bounds the random parameters are described by their first- and second-order moments only. The report starts by outlining the theory and follows by describing two possible applications: one for the derivation of the scattering function of a moving, turning line target; the other in the use of pulse trains to measure scattering functions.

INTRODUCTION

This report is the sequel to an earlier report <1> on linear, time-varying systems theory and its application to active-sonar signal processing. The major concern in that report was with deterministic systems, while the main concern in this report is with stochastic systems. The main results of the earlier report may be summarized as follows: a linear, time-varying system may be viewed as spreading a signal in time and either frequency or time stretch (doppler); therefore it is convenient to represent such systems by a function of time shift τ and frequency shift ν or time stretch α known as the spreading function S or S^D and to represent signals by another function of τ and ν or α known as the cross-ambiguity function ϕ or ϕ^D. The cross-ambiguity function of the output of such a system is a modified double convolution in τ and ν or α between its spreading function and the cross-ambiguity function of the input; furthermore the spreading function of the concatenation of two such systems is the same modified double convolution of their spreading functions.

In this report we are concerned with linear, time-varying systems for which $S(\tau,\nu)$ or $S^D(\tau,\alpha)$ are random variables; such systems are then described by their statistical properties. The general range of statistical behaviour possible is far too vast for this brief report; therefore we must narrow our concern by the use of assumptions that vastly simplify the analysis while only marginally restricting its application. It is quite reasonable to assume that the systems in which we are interested are statistically independent of one another and their inputs. With this assumption it is also quite reasonable to assume that the means $E[S(\tau,\nu)]$ or $E[S^D(\tau,\alpha)]$ are zero (where $E[\ ]$ signifies expected value of) since the effect of non-zero means may be analyzed separately with the deterministic theory of <1>. Since we are dealing with linear systems, it is also reasonable to restrict our concern to second order statistics -- the variance of $S(\tau,\nu)$ and $S^D(\tau,\alpha)$. Finally we assume the values of $S(\tau,\nu)$ or $S^D(\tau,\alpha)$ are uncorrelated for differing τ and ν or α; hence the second-order statistics of S of S^D can be represented by the functions $\$(\tau,\nu)$ or $\$^D(\tau,\alpha)$, known as the scattering function, which is the analogue of the power density of a white-noise process:

$$E[S^*(\tau',\nu')\, S(\tau,\nu)] = \$(\tau,\nu)\,\delta(\tau'-\tau)\,\delta(\nu'-\nu) \;,$$

$$E[S^{D*}(\tau',\alpha')\, S(\tau,\alpha)] = \$^D(\tau,\alpha)\,\delta(\tau'-\tau)\,\delta(\alpha'-\alpha) \;, \qquad \text{(Eq. 1)}$$

(where $*$ represents complex conjugate and $\delta(\ )$ is the Dirac delta function).

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The transmitted signal is assumed to be deterministic; hence its ambiguity function is real. On the other hand, once the signal has passed through a stochastic system of the type discussed in the last paragraph it becomes stochastic with zero mean; therefore in our stochastic theory signals will be represented by power cross-ambiguity functions $E[|\phi|^2]$ or $E[|\phi^D|^2]$.

The fundamental result of import is that the power cross-ambiguity function of the output of a system describable by a scattering function is a modified convolution in τ and ν or α between its scattering function and the input power cross-ambiguity function; furthermore the scattering function of the concatenation of two systems describable by scattering functions is the same modified convolution in τ and ν or α between their scattering functions. In fact, the required modified double convolution in τ and ν is just a double convolution, while the required modified double convolution in τ and α is the same modified double convolution required by the deterministic theory and discussed in detail in <1>. These results are illustrated in Figs. 1 and 2; the frequency shift version of the results is well known <2>, but the time stretch (doppler) version is believed to be novel.

The fact that in the frequency-shift version of the theory an ordinary double convolution is involved should not be a big surprise. The assumption that Eq. 1 holds for $S(\tau,\nu)$ implies that $E[h(t',\tau')\ h(t,\tau)]$ is non-zero only for $\tau' = \tau$ and depends only on $t'-t$; that is, that the time-varying impulse response $h(t,\tau)$ consists of a statistically wide sense stationary set of uncorrelated scatterers. Often this assumption is abbreviated WSSUS for wide sense stationary uncorrelated scatterers. The implication of this result is that, unlike the deterministic case and the time-stretch version of the stochastic case, the order in which systems are concatenated is unimportant. These results are exactly analogous to the results of time-invariant linear system theory in which ordinary convolutions are used and the order of concatenation is unimportant. Unfortunately, while the frequency-shift scattering function is a good model of the medium, to correctly represent a thin, rigid, moving, turning target (such as a submarine) the time-stretch scattering function is required (and even this is an approximation). For unity BT signals such as CW signals, such a target may be represented approximately by a frequency-shift scattering function derived in an obvious way from the time stretch scattering function, but for high BT signals such as linear FM signals the time-stretch version must be used.

In many cases it is desirable to try to measure scattering functions. For example, knowing the scattering function of a medium tells us the statistics of how it spreads in time and frequency a signal transmitted through it, and knowing the scattering function of a target allows us to measure interesting target properties <3>. Thus the design of a suitable signal for measuring scattering functions is of considerable interest and has been the subject of considerable studies, for example by Rihaczek <4>. The rub is that the volume under the power ambiguity function $|\gamma|^2$ is constrained to a given value, so that it is impossible to determine the scattering function exactly. Costa and Hug <5> suggested the use of pulse trains to measure the under-spread scattering functions (those whose extent in time and frequency has a product less than unity). A pulse train has a sharp spike at the origin, with the remainder of its volume spead outside a region whose area is unity. For an evenly spaced train of identical pulses

this volume consists of respective spikes in both the time and frequency shift directions, but these may be modified by any number of tricks such as are given in <4>. In their work Hug and Costa make implicit use of an approximate method of realizing the bank of matched filters required to determine the ordinary output power cross-ambiguity function that can most simply be described as the temporal analogue of narrowband (phase-shift) beamforming. In this report a similar technique is described for realizing the bank of matched filters required to determine the time-stretch output power cross-ambiguity function that is the temporal analogue of broadband (time-shift) beamforming.

This report is divided into two parts covering theory and applications respectively. The theoretical part (Ch. 1) is largely a detailed presentation of the basic relationship given in Figs. 1 and 2, while the part on application (Ch. 2) derives the time-shift scattering function of a moving, turning, line target and investigates the use of coherent pulse trains in measuring target scattering functions. Detailed derivations are relegated to the appendices.

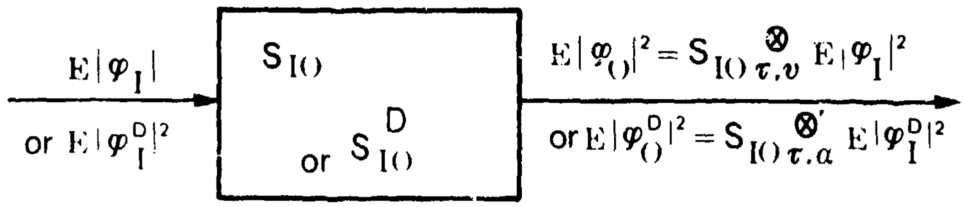

FIG. 1 BASIC INPUT-OUTPUT RELATIONSHIP

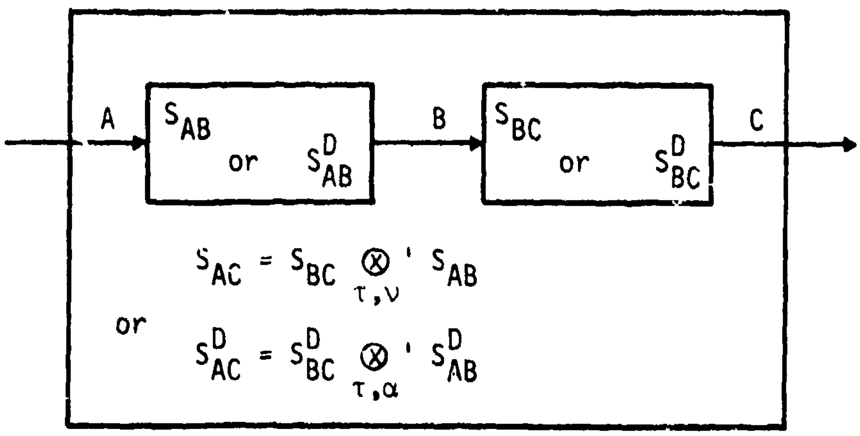

FIG. 2 BASIC CONCATENATION RELATIONSHIPS

1 THEORY

This chapter consists of two parts: discussion on scattering functions (Sect. 1.1) and discussion of the fundamental relationships (Sect. 1.2).

1.1 Scattering functions

The scattering function of a stochastic, linear, time-varying system was given in Eq. 1 in terms of the spreading function of such a system, where the spreading function is defined in detail in <1>. Not all stochastic, linear, time-varying systems have scattering functions, of course, since a number of assumptions, which were enumerated in the introduction, have to be satisfied; furthermore, if a system has a frequency-shift scattering function it will not have a time-stretch scattering function and vice-versa. Nevertheless, many important systems come close enough to obeying the required assumptions that they can be represented by scattering functions and in some situations a system with a time-stretch scattering function may be represented by a frequency-shift scattering function.

Recall from <1> that the impulse response of a linear, time-varying system is related to its spreading function via

$$h(t,\tau) = \int_{-\infty}^{\infty} e^{2\pi j\nu(t-\tau)}\, S(\tau,\nu)\, d\nu \; ; \qquad \text{(Eq. 2)}$$

therefore if the system has a scattering function

$$E[h^*(t',\tau')h(t,\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j[\nu(t-\tau)-\nu'(t'-\tau')]}E[S^*(\tau',\nu')S(\tau,\nu)]d\nu\, d\nu'$$

$$= \int_{-\infty}^{\infty} e^{2\pi j\nu(t-t')}\$(\tau,\nu)d\nu\, \delta(\tau'-\tau) \qquad \text{(Eq. 3)}$$

Thus $h(t,\tau)$ consists of uncorrelated scatters whose statistics are time invariant. Note the presence in Eq. 3 of the inverse Fourier transform of the scattering function. Just as in <1> there was no simple relationship between $S^D(\tau,\nu)$ and $h(t,\tau)$, there is no simple relationship between the statistics of $h(t, \tau)$ and $\$^D(\tau,\alpha)$ for systems having a time-stretch scattering function.

1.2 The fundamental relationships

Consider the two operators $\underset{\tau,\nu}{\otimes}$ and $\underset{\tau,\alpha}{\otimes}'$ defined by

$$b \underset{\tau,\nu}{\otimes} a(\tau,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\tau-\tau', \nu-\nu')\, b(\tau',\nu')\, d\nu'\, d\tau' \qquad \text{(Eq. 4a)}$$

$$b^D \underset{\tau,\alpha}{\otimes} a^D(\tau,\alpha) = \int_{-\infty}^{\infty}\int_{0}^{\infty} a^D[\alpha'(\tau-\tau'), \tfrac{\alpha}{\alpha'}]\, b^D(\tau',\alpha')\, d\alpha'\, d\tau' \qquad \text{(Eq. 4b)}$$

The former is nothing more than the well-known double convolution operator, which is both communicative and associative. The latter is the modified double-convolution operator studied in detail in <1>, where it was shown to be associative but not communicative. Use of these definitions, those of Eq. 1 and the fundamental input-output and concatenation relationships for spreading and cross-ambiguity function given in <1> readily yields the fundamental relationships described in Figs. 1 and 2 – details are contained in Appendix A.

2 APPLICATIONS

Two applications of the theory of Ch. 1 are presented here: derivation of the scattering function of a moving, turning, line target, and the use of pulse trains to measure scattering functions. Discussion of the latter topic is broken in turn into two parts: derivation of its power ambiguity function, and presentation of an approximate method of simply realizing the matched filter bank when a pulse train is used.

2.1 The doppler scattering function of a moving, turning, line target

From Eq. 22 of <1> the doppler spreading function of a moving, turning target is

$$S^D(\tau,\alpha) \cong \frac{c}{(1-a)\delta}\, f\left(\frac{-c\tau+\ell-\frac{\alpha-1}{\alpha}\, r_T}{(1-a)\,\delta}\right) \delta\left[\alpha-1 - \ell/c - \frac{\beta}{1-a}\left(\tau_\xi - \frac{\ell}{c} - \frac{\beta}{1-a}\right)\left(\tau - \frac{\ell}{c} + \frac{\alpha-1}{\alpha}\,\frac{r_T}{c}\right)\right] ,$$

(Eq. 5)

where $f(\sigma)$ is the reflectivity of the point along the target located at distance σ from the centre. The remaining quantities in Eq. 5 are defined in <1> and since they are not of direct concern their definition is not repeated here. Now we make the assumption that

$$E[f^*(\sigma')f(\sigma)] = R_f(\sigma)\,\delta(\sigma'-\sigma) \qquad \text{(Eq. 6)}$$

For our theory to be valid it is not necessary that Eq. 6 hold exactly, but only that the reflectivity of points on the target be uncorrelated at distances greater than the distance resolution of the transmitted signal -- a condition that is usually met in practice.

Now from Eqs. 5 and 6 we have

$$\begin{aligned}
E\left[S^{D*}(\tau',\alpha')S^D(\tau,\alpha)\right] &\cong \frac{c}{(1-a)\delta}\, R_f\left(\frac{-c\tau+\ell-\frac{\alpha-1}{\alpha}\, r_T}{(1-a)\,\delta}\right)\, \delta\left[\tau'-\tau+\left(\frac{\alpha'-1}{\alpha'} - \frac{\alpha-1}{\alpha}\right)\right] \\
&\quad \cdot \delta\left[\alpha-1 + \frac{\dot{\ell}}{c} - \frac{\beta}{1-a}\left(\tau - \frac{\ell}{c} + \frac{\alpha-1}{\alpha}\,\frac{r_T}{c}\right)\right] \\
&\quad \cdot \delta\left[\alpha'-1 + \frac{\dot{\ell}}{c} - \frac{\beta}{1-a}\left(\tau' - \frac{\ell}{c} + \frac{\alpha'-1}{\alpha'}\,\frac{r_T}{c}\right)\right] \\
&= \frac{c}{(1-a)\delta}\, R_f\left(\frac{-c\tau+\ell-\frac{\alpha-1}{\alpha}\, r_T}{(1-a)\delta}\right)\delta\left[\alpha-1 + \frac{\dot{\ell}}{c} - \frac{\beta}{1-a}\left(\tau - \frac{\ell}{c} + \frac{\alpha-1}{\alpha}\,\frac{r_T}{c}\right)\right] \\
&\quad \cdot \delta(\tau'-\tau)\;\delta(\alpha'-\alpha) ,
\end{aligned}$$

(Eq. 7)

making use of identities implied by the delta function. Obviously from Eq. 1 and this result, the doppler scattering function is:

$$\$^{D}(\tau,\alpha) \cong \frac{c}{(1-a)f} R_f\left(\frac{-c\tau + \ell - \frac{\alpha-1}{\alpha} r_T}{(1-a)\delta}\right) \delta\left[\alpha-1 + \frac{\dot{\ell}}{c} - \frac{\beta}{1-a}\left(\tau - \frac{\ell}{c} + \frac{\alpha-1}{\alpha}\frac{r_T}{c}\right)\right] , \quad \text{(Eq. 8)}$$

which has the same form as that of the spreading function as given in Eq. 5.

2.2 The power ambiguity function for a pulse train

In this and the following section we are concerned with an evenly spaced train of identical pulses:

$$\chi_{PT}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \chi_p[t - (i - \frac{n+1}{2}) \Delta T] \quad \text{(Eq. 9)}$$

where the pulse $\chi_p(t)$ is assumed to be normalized to have unity total energy (i.e. the integral of $|\chi_p|^2$ over t is unity).
Rihaczek <4> has already determined the ordinary ambiguity function $\gamma_{PT}(\tau,\nu)$ for such a signal to be

$$\gamma_{PT}(\tau,\nu) = \sum_{i=-n+1}^{n-1} \gamma_i(\tau - i\,\Delta T, \nu) , \quad \text{(Eq. 10)}$$

where

$$\gamma_i(\tau,\nu) = e^{-i\pi j\nu\Delta T} \frac{\sin[\pi\nu(n-|i|\Delta T]}{n \sin(\pi\nu\Delta T)} \gamma_p(\tau,\nu) \quad \text{(Eq. 11)}$$

and $\gamma_p(\tau,\nu)$ is the ordinary ambiguity function of $\chi_p(t)$.

If we assume — as is typically the case — that the spread of γ_p in τ is small compared with Δt, the ordinary power ambiguity function $|\gamma_{PT}(\tau,\nu)|^2$ for the pulse train is

$$|\gamma_{PT}(\tau,\nu)|^2 = \sum_{i=-n+1}^{n-1} |\gamma_i(\tau,\nu)|^2 , \quad \text{(Eq. 12)}$$

where

$$|\gamma_i(\tau,\nu)|^2 = \frac{\sin[\pi\nu(n-|i|)\Delta T]}{n\sin(\pi\nu\Delta T)}^2 \; |\gamma_p(\tau,\nu)|^2 \qquad \text{(Eq. 13)}$$

Except for $|i| = n$, the first factor on the right side of Eq. 13 is a function of ν that is a series of spikes that are $(n-|i|)/n$ tall, roughly $1/[(n-i)\Delta T]$ wide, and separated by $1/\Delta T$. Since the width of $|\gamma_p|^2$ is small compared with ΔT, $|\gamma_{PT}|^2$ is thus a pattern of spikes in (τ,ν) space. This pattern is discussed and illustrated in detail in <4>.

Now consider the doppler power ambiguity function of the pulse train. In Appendix B it is shown, by substitution of Eq. 9 into the definition of the doppler ambiguity function given in <1>, that

$$\gamma^D_{PT}(\tau,\alpha) = \sum_{i=-n+1}^{n-1} \gamma^D_i(\tau - i\,\Delta T,\, \alpha) \quad , \qquad \text{(Eq. 14)}$$

where

$$\gamma^D_i(\tau,\alpha) = \frac{1}{n}\sum_{k=1}^{n-|i|} \gamma^D_p\left(\tau + \frac{\alpha-1}{\alpha}\left(\frac{n+1}{2} - k + \frac{i-|i|}{2}\right)\Delta T,\ \alpha\right) \qquad \text{(Eq. 15)}$$

and $\gamma^D_p(\tau,\alpha)$ is the doppler ambiguity function $\chi_p(t)$. The analogy between Eqs. 10 and 14 is immediate; Eqs. 11 and 15 are best compared through Fourier transforms, which are obvious for γ^D and derived in Appendix B for γ^D_i:

$$\Gamma_i(f,\nu) = e^{-i\pi j\nu\Delta T}\left(\frac{\sin[\pi\nu(n-|i|)\Delta T]}{n\sin(\pi\nu\Delta T)}\right)\Gamma_p(f,\nu) \qquad \text{(Eq. 16)}$$

$$\Gamma^D_i(f,\alpha) = e^{-i\pi j\frac{\alpha-1}{\alpha}f\Delta T}\left(\frac{\sin\left[\pi\frac{\alpha-1}{\alpha}f(n-|i|)\Delta T\right]}{n\sin\left(\pi\frac{\alpha-1}{\alpha}f\,\Delta T\right)}\right)\Gamma^D_p(f,\alpha) \quad , \qquad \text{(Eq. 17)}$$

where Γ_p and Γ^D_p are the Fourier transforms with respect to τ of γ_p and γ^D_p respectively.

Again we assume that the spread of γ_p^D in τ is small compared with ΔT. In this case, the doppler power ambiguity function $|\gamma_{PT}^D(\tau,\alpha)|^2$ for the pulse train is

$$|\gamma_{PT}(\tau,\nu)|^2 = \sum_{i=-n+1}^{n-1} |\gamma_i^D(\tau,\nu)|^2 \qquad \text{(Eq. 18)}$$

for $|\alpha-1|/\alpha < 1/n$. This constraint, which follows direct from Eq. 15 and is derived in Appendix B, insures that the γ_i^D do not overlap significantly. The analogue of Eq. 13 is not given since the magnitude squared of Eq. 15 does not take a simple form.

Instead we determine the total energy in $\gamma_i^D(\tau,\alpha)$ for a given α by integrating $|\gamma_i^D(\tau,\alpha)|^2$ over τ or equivalently from Parseval's relationship by integrating $|\Gamma_i^D(f,\alpha)|^2$ over f. If a similar procedure is carried out for Eq. 16 the result is $(\sin[\pi\nu(n-|i|)\Delta T]/[n\sin(\pi\nu\Delta T)])^2$ multiplied by the energy in $\gamma_p'(\tau,\nu)$ for a given ν. For a bandpass pulse, $\Gamma_p^D(f,\alpha)$ will be centred more or less around $f = f_o$, the carrier frequency; therefore the result of integrating $|\Gamma_i^D(f,\alpha)|^2$ is

$$\{\sin[\pi \tfrac{\alpha-1}{\alpha} f_o(n-|i|)\Delta t]/[n\sin(\pi \tfrac{\alpha-1}{\alpha} f_o\Delta T)]\}^2$$

smoothed over α, multiplied by the energy in $\gamma_p^D(\tau,\alpha)$ for a given α. Thus $|\gamma_{PT}^D|^2$ is a pattern of spikes in (τ,ν) space analogous to the pattern of spikes in (τ,ν) space of $|\gamma_{PT}|^2$; however, since the smoothing effect just mentioned increases linearly with α, the spikes become lower and broader with increasing α.

To illustrate the ideas of the previous paragraph, an example will be presented. Let $x_p(t)$ be an FM pulse with centre frequency f_o and bandwidth B. From <1>

$$|\Gamma_{FM}^D(f,\alpha)|^2 \approx \frac{1}{B} \left[f - \frac{\alpha+1}{2} f_o ; (B - |\alpha-1| f_o)/2\right] , \qquad \text{(Eq. 19)}$$

where

$$⎍(t;T) = \begin{cases} 1 & |t| \leqq T \\ 0 & \text{otherwise} \end{cases} ;$$

therefore

$$\int_{-\infty}^{\infty} |\Gamma_i^D(f,\alpha)|^2 \, df = \frac{1}{B} \int_{-(B-|\alpha-1|f_0)/2}^{(B-|\alpha-1|f_0)/2} \left(\frac{\sin\left[\pi \frac{\alpha-1}{\alpha}(f+\frac{\alpha+1}{2}f_0)(n-|i|\Delta T)\right]}{n \sin\left[\pi \frac{\alpha-1}{\alpha}(f+\frac{\alpha+1}{2} f_0)\Delta T\right]} \right)^2 df$$

(Eq. 20)

$$= \frac{1}{\mu} \int_{-(\mu-|\alpha-1|)/2}^{(\mu-|\alpha-1|)/2} \left(\frac{\sin\left[\pi\xi \, x+\frac{\alpha+1}{2})(n-|i|)\right]}{n \sin\left[\pi\xi(x+\frac{\alpha+1}{2})\right]} \right)^2 dx \quad ,$$

where

$$\xi = \frac{\alpha-1}{\alpha} f_o \, \Delta T \quad , \qquad \mu = \frac{B}{f_o} \quad .$$

The integral of Eq. 20 is readily performed since the integrand can be written as a sum of cosine functions. On the other hand for the ordinary power ambiguity function we have

$$\int_{-\infty}^{\infty} |\Gamma_i(f,\nu)|^2 \, df = \frac{B-\nu}{B} \left(\frac{\sin[\pi\nu(n-|i|)\Delta T]}{n \sin(\pi\nu\Delta T)} \right)^2$$

$$= (1-\frac{\nu}{\mu f_0}) \left(\frac{\sin[\pi\xi(n-|i|)]}{n \sin(\pi\xi)} \right)^2 \qquad \text{(Eq. 21)}$$

where $\xi = \nu \Delta T$ and μ is as before. Graphs of these two functions versus ξ are given in Fig. 3 for $\mu = 0.06$, $|\alpha-1|$, and $\nu/f_o \ll \mu$, $i = 0$, and $n = 2$, 3 and 5. These graphs bear out the conclusions of the previous paragraph.

From the foregoing it is apparent that evenly spaced trains of identical pulses have ambiguity functions with a central spike surrounded by an essentially clear region of unity area; hence by appropriate choice of signal parameters it is possible to estimate a scattering function known to be contained in region of unity area.

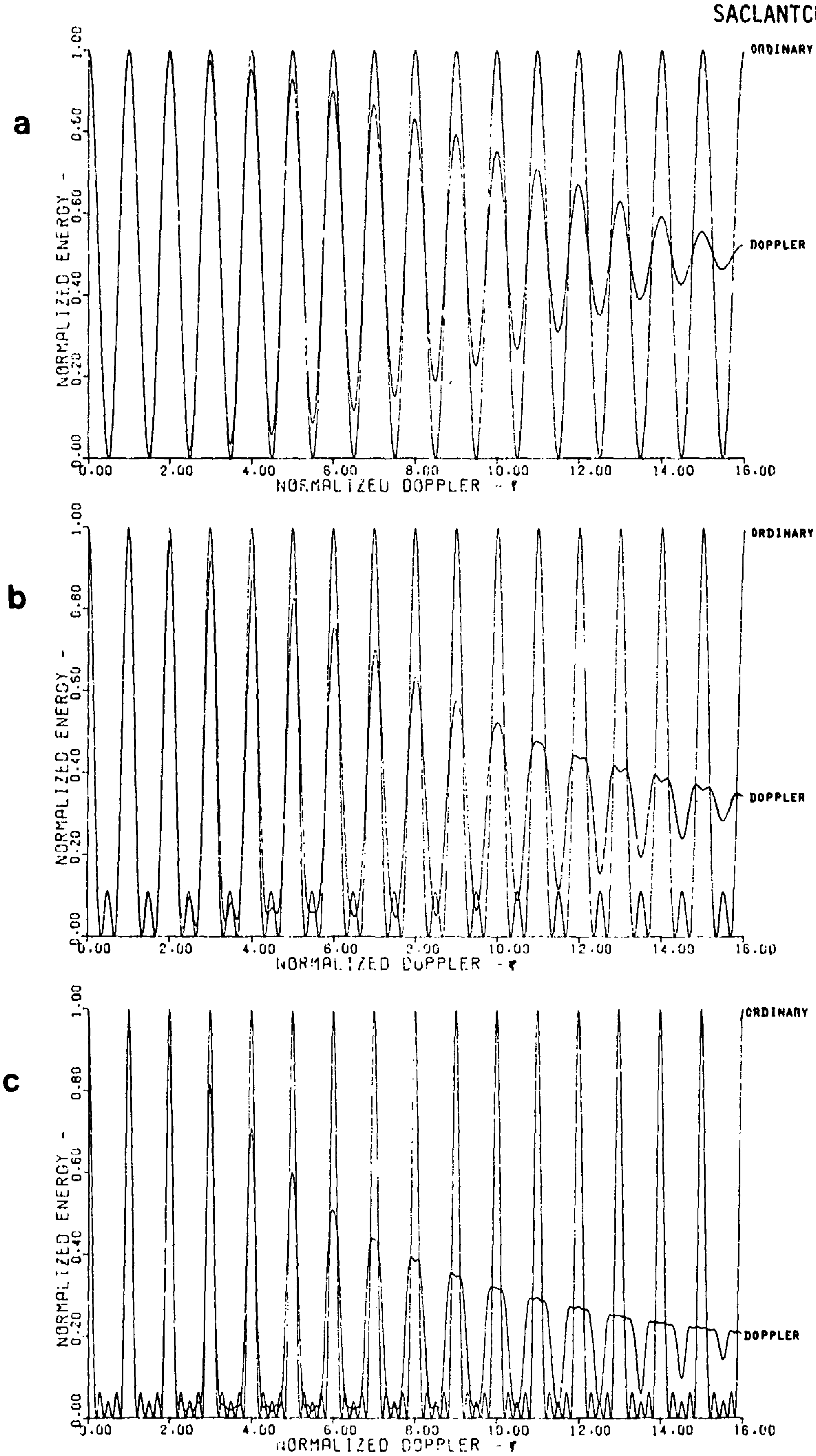

FIG. 3 NORMALIZED ENERGY IN AMBIGUITY FUNCTION FOR $i = 0$

a) $n = 2$ *b)* $n = 3$ *c)* $n = 5$

The cost of such capability is ambiguity about the absolute position of the scattering function. In many cases – for example determination of medium scattering functions – this ambiguity is unimportant since it may be resolved by other information. It is possible, as Rihaczek <4> discusses at length, to modify an evenly spaced train of pulses in such a manner that the non-central spikes of the ambiguity function are lowered and spread out. Without going into details, the possibilities are as follows: By shifting the phase of the individual pulses or, better yet, making the spacing between pulses non-uniform γ_i or γ_i^D may be lowered and spread in τ – thus alleviating the ambiguity in τ. By, in addition, making the pulses non-identical – for example stepping them in frequency – it is possible to generate a signal with a "thumb tack" ambiguity function (i.e. a central spike surrounded by a more or less uniform plateau). The cost of the latter step is that the simplified processing for pulse trains of identical pulses to be described next cannot be used.

2.3 An approximation to the matched filter bank for a pulse train

For ease of presentation a uniformly spaced train of identical pulses is assumed; however, the same analysis applies in an obvious manner if the pulses are shifted in time or phase. Furthermore, it is assumed that the individual pulses are FM pulses; however, the same analysis again applies in an obvious manner to any wide band pulses.

Let $y(t)$ be the signal to be passed through the bank of filters matched to $\chi_{PT}(t)$ and frequency-shifted versions of $\chi_{PT}(t)$. From Eq. 9 and the approximate identity

$$e^{2\pi j\nu t}\chi_{FM}(t) \approx e^{-2\pi j\frac{f_o\nu}{k}}\chi_{FM}\left(t+\frac{\nu}{k}\right), \qquad \text{(Eq. 22)}$$

where f_o is the centre frequency and k the frequency slope of $\chi_{FM}(t)$, it is shown in Appendix C by substitution that the output of the bank of matched filters is

$$\phi_{PT}(\tau,\nu) \triangleq \int_{-\infty}^{\infty} e^{-2\pi j\nu(t-\tau)}\chi^*_{PT}(t-\tau)\,y(t)\,dt$$

$$\approx \frac{e^{2\pi j\frac{f_o\nu}{k}}}{\sqrt{n}}\sum_{i=1}^{n} e^{-2\pi j\nu\left(i-\frac{n-1}{2}\right)\Delta T}\,\phi_{FM}\left[\tau-\frac{\nu}{k}+\left(i-\frac{n+1}{2}\right)\Delta T,\ o\right], \qquad \text{(Eq. 23)}$$

where

$$\phi_{FM}(\tau,o) \triangleq \int_{-\infty}^{\infty} x_{FM}^{*}(t-\tau)\, y(t)\, dt \qquad \text{(Eq. 24)}$$

Now consider a bank of filters matched to $x_{PT}(t)$ and doppler-shifted versions of $x_{PT}(t)$. From Eq. 9 and the approximate identity

$$x_{FM}(\alpha t) \approx e^{-2\pi j \frac{(\alpha-1)f_o^2}{k}} x_{FM}[t + (\alpha-1)\, f_o/k] \quad , \qquad \text{(Eq. 25)}$$

it is shown in Appendix C by substitution that the output of the bank of matched filters is

$$\phi_{PT}^{D}(\tau,\alpha) \triangleq \int_{-\infty}^{\infty} x_{PT}^{*}[\alpha(t-\tau)]y(t)\, dt$$

$$\approx \frac{e^{2\pi j \frac{(\alpha-1)f_o^2}{k}}}{\sqrt{n}} \sum_{i=1}^{n} \phi_{FM} [\tau - \frac{(\alpha-1)\, f_o}{k} + \frac{1}{\alpha} (i - \frac{n+1}{2})\, \Delta T,\, o].$$

(Eq. 26)

The quantity $\phi_{FM}(\tau,o)$ is nothing more than the output for a filter matched to the FM pulse $x_{FM}(t)$; therefore the first step in the approximate realization of either the frequency-shift or doppler-shift bank of matched filters is to pass the signal through a filter matched to the transmitted pulse. For a given τ , Eq. 23 for the frequency-shift bank of matched filters takes - except for a time and phase shift - the form of a discrete Fourier transform of the sequence

$$\phi_{FM}[\tau+(i - \frac{n+1}{2})\, \Delta T,\, o] \ , \quad i = 1, \ldots, n$$

of outputs from the initial matched filter. Similarly for a given τ , Eq. 26 for the doppler-shift bank of matched filters takes - except for a time and phase shift - a form similar to the discrete Fourier transform of y_{FM} except that time stretch by $1/\alpha$ replaces frequency shift by $-v$ (as might

be expected). In fact if $y_{FM}[\tau-(1-\frac{n-1}{2})\Delta T, o]$ is identified as the matched filter output for a hydrophone located at point $(i-\frac{n-1}{2})\Delta x$ along a linear array, Eqs. 23 and 26, with Δt replaced by Δx, take the form of a narrow-band beam former and a broadband beamformer respectively.
From this correspondence it is clear that Fig. 3 gives narrow-band and broadband beam patterns for 2, 3 and 5 element arrays where the abscissa is the cosine of the bearing angle relative to the antenna suitably normalized.

CONCLUSIONS

Under appropriate conditions random linear time-varying systems may be represented by a function of time shift and frequency shift or time stretch known as the scattering function and the ouptut of such systems by a second function of the same variables known as the power cross-ambiguity function. The input-output and concatenation relationships for such systems are double convolutions in the frequency-shift version and modified double convolutions in the time-stretch (doppler) version. A moving, turning line target has a doppler scattering function that is non-zero only along a line in time-shift, time-stretch space. A very useful signal to estimate scattering functions such as those of a medium or a target is the pulse train, because its ambiguity function has a spike at the origin surrounded by a clear region of unit area. The matched filter bank for such a signal when the individual pulses are identical and wide band consists of a filter matched to the individual pulse followed by the temporal analogue of a narrow-band or broadband beam former.

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APPENDICES

APPENDIX A

DERIVATION OF THE FUNDAMENTAL RELATIONSHIPS

From <A.1>

$$\phi_O(\tau,\nu) = S_{IO} \underset{\tau,\nu}{\otimes}{}' \; \phi_I(\tau,\nu)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi j\nu'(\tau-\tau')} \phi_I(\tau-\tau',\nu-\nu') \, S_{IO}(\tau',\nu') \, d\nu' \, d\tau' \quad ;$$

therefore, if Eq. 1 of the main text holds for S_{IO}

$$E\left[|\phi_O(\tau,\nu)|^2\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi j[\nu'(\tau-\tau')-\nu''(\tau-\tau'')]} E[\phi_I^*(\tau-\tau'',\nu-\nu'')\phi_I(\tau-\tau',\nu-\nu')]$$

$$\cdot \, E\,[S_{IO}^*(\tau'',\nu'') \, S_{IO}(\tau',\nu')] \, d\nu'' \, d\tau'' \, d\nu' \, d\tau'$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\,[\,|\phi_I(\tau-\tau',\nu-\nu')|^2] \, \$_{IO}(\tau',\nu') \, d\nu' \, d\tau'$$

$$= \$_{IO} \underset{\tau,\nu}{\otimes} E[|\phi_I(\tau,\nu)|]^2 \quad .$$

Similarly from <A.1>

$$\phi_O^D(\tau,\alpha) = S_{IO}^D \underset{\tau,\alpha}{\otimes}{}' \; \phi_I^D(\tau,\alpha)$$

$$= \int_{-\infty}^{\infty} \int_{0}^{\infty} \phi_I^D[\alpha'(\tau-\tau'), \tfrac{\alpha}{\alpha'}] \, S_{IO}^D(\tau',\alpha') \, d\alpha' \, d\tau' \quad ;$$

therefore if Eq. 1 of the main text holds for S_{IO}^{D}

$$E\left[|\phi_O(\tau,\nu)|^2\right] = \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} E\left\{\phi_I^{D*}\left[\alpha''(\tau-\tau''),\frac{\alpha}{\alpha''}\right]\phi_I^{D}\left[\alpha'(\tau-\tau'),\frac{\alpha}{\alpha'}\right]\right\}$$

$$\cdot\, E\left[S_{IO}^{D*}(\tau'',\alpha'')\, S_{IO}^{D}(\tau',\alpha')\right] d\alpha''\, d\tau''\, d\alpha'\, d\tau'$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} E\left\{|\phi_I^{D}\left[\alpha'(\tau-\tau'),\frac{\alpha}{\alpha'}\right]|^2\right\} \$^{D}(\tau',\alpha')\, d\alpha'\, d\tau'$$

$$= \$_{IO}^{D} \underset{\tau,\alpha}{\otimes}{}'\; E\,|\phi_I^{D}(\tau,\alpha)|^2 \quad .$$

Thus the fundamental relationships of Fig. 1 of the main text are valid.

Also from <A.1>

$$S_{AC}(\tau,\nu) = S_{BC} \underset{\tau,\nu}{\otimes}{}'\; S_{AB}(\tau,\nu)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j\nu'(\tau-\tau')}\, S_{AB}(\tau-\tau',\nu-\nu')\, S_{BC}(\tau',\nu')\, d\nu'\, d\tau' \quad ;$$

therefore, if Eq. 1 of the main text holds for S_{AB} and S_{BC}

$$E\left[S_{AC}^{*}(\tau,\nu)\, S_{AC}(\tau',\nu')\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi j[\nu''(\tau-\tau'')-\nu'''(\tau'-\tau''')]}$$

$$\cdot\, E\left[S_{AB}^{*}(\tau'-\tau''',\nu'-\nu''')S_{AB}(\tau-\tau'',\nu-\nu'')\right] E\left[S_{BC}^{*}(\tau''',\nu''')S_{BC}(\tau'',\nu'')\right]$$

$$\cdot\, d\nu'''d\tau'''d\nu''d\tau''$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \$_{AB}(\tau-\tau'',\nu-\nu'')\, \$_{BC}(\tau'',\nu'')\, d\nu''\, d\tau''\, \delta(\tau-\tau')\delta(\nu-\nu')$$

$$= \$_{BC} \underset{\tau,\nu}{\otimes} \$_{AB}(\tau,\nu)\delta(\tau-\tau')\delta(\nu-\nu')$$

and Eq. 1 of the main text holds for $\$_{AC}$. Similarly from <A.1>

$$S^{D}_{AC}(\tau,\alpha) = S^{D}_{BC} \underset{\tau,\alpha}{\otimes}{}' \; S^{D}_{AB}(\tau,\alpha)$$

$$= \int_{-\infty}^{\infty} \int_{0}^{\infty} S^{D}_{AB}\left[\alpha'(\tau-\tau'),\, \frac{\alpha}{\alpha'}\right] S^{D}_{BC}(\tau',\alpha')\, d\alpha'\, d\tau' \quad ;$$

therefore if Eq. 1 of the main text holds for S^{D}_{AB} and S^{D}_{BC}

$$E\left[S^{D*}_{AC}(\tau,\nu)\; S^{D}_{AC}(\tau',\nu')\right] = \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} E\left\{S^{D*}_{AB}\left[\alpha'''(\tau'-\tau'''),\frac{\alpha}{\alpha'''}\right] S^{D}_{AB}\left[\alpha''(\tau-\tau''),\frac{\alpha}{\alpha''}\right]\right\}$$

$$\cdot\; E\left[S^{D*}_{BC}(\tau''',\alpha''')S^{D}_{BC}(\tau'',\nu'')\right] d\alpha'''\; d\tau'''\; d\alpha''\; d\tau''$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} \$^{D}_{AB}\left[\alpha''(\tau-\tau''),\frac{\alpha}{\alpha''}\right] \$^{D}_{BC}(\tau'',\nu'')d\alpha''\; d\tau''\delta(\tau-\tau')\delta(\alpha-\alpha')$$

$$= \$^{D}_{BC} \underset{\tau,\alpha}{\otimes}{}' \; \$^{D}_{AB}(\tau,\alpha)\; \delta(\tau-\tau')\delta(\alpha-\alpha')$$

and Eq. 1 of the main text holds for $\$^{D}_{AC}$ Thus the fundamental relationships of Fig. 2 of the main text are valid.

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APPENDIX B

DERIVATION OF THE DOPPLER AMBIGUITY FUNCTION OF A UNIFORM TRAIN OF IDENTICAL PULSES AND SOME OF ITS PROPERTIES

By definition (<B.1>, Eq. 1b):

$$\gamma^{D}_{pT}(\tau,\alpha) \triangleq \int_{-\infty}^{\infty} \chi^{*}_{pT}\left[\alpha(t-\tau)\right]\chi(t)\, dt$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left\{\sum_{k=1}^{n}\int_{-\infty}^{\infty}\chi^{*}_{p}\left[\alpha(t-\tau)-\left(k-\frac{n+1}{2}\right)\Delta T\right]\chi_{p}\left[t-\left(i-\frac{n+1}{2}\right)\Delta T\right] dt\right\}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\sum_{k=1}^{n}\int_{-\infty}^{\infty}\chi^{*}_{p}\left(\alpha\left\{t-\tau+\left[i-\frac{k}{\alpha}-\frac{\alpha-1}{\alpha}\left(\frac{n+1}{2}\right)\right]\Delta T\right\}\right)\chi_{p}(t)\, dt\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\gamma^{D}_{p}\left\{\tau-\left[i-k-\frac{\alpha-1}{\alpha}\left(\frac{n+1}{2}-k\right)\right]\Delta T\ ,\ \alpha\right\}\right)$$

$$= \sum_{i=-n+1}^{n-1}\left(\frac{1}{n}\sum_{k=1-\frac{1-|i|}{2}}^{n-\frac{1+|i|}{2}}\gamma^{D}_{p}\left\{\tau-\left[i-\frac{\alpha-1}{\alpha}\left(\frac{n+1}{2}-k\right)\right]\Delta T,\ \alpha\right\}\right)$$

$$= \sum_{i=-n+1}^{n-1}\left(\frac{1}{n}\sum_{k=1}^{n-|i|}\gamma^{D}_{p}\left\{\tau-\left[i-\frac{\alpha-1}{\alpha}\left(\frac{n+1}{2}-k+\frac{i-|i|}{2}\right)\right]\Delta T,\ \alpha\right\}\right).$$

Substitution of Eq. 15 of the main text into this result yields Eq. 14 of the main text.

The Fourier transform of Eq. 15 of the main text is by definition

$$\Gamma_i^D(f,\alpha) \triangleq \mathcal{F}\left[\gamma_i^D(\tau,\alpha)\right]$$

$$= \frac{1}{n}\sum_{k=1}^{n-|i|} \mathcal{F}\left\{\gamma_p^D\left[\tau+\frac{\alpha-1}{\alpha}\left(\frac{n+1}{2} - k + \frac{i-|i|}{2}\right)\Delta T,\ \alpha\right]\right\}$$

$$= \frac{1}{n}\sum_{k=1}^{n-|i|}\left\{e^{-2\pi j\,\frac{\alpha-1}{\alpha}\,f\left(\frac{n+1}{2} - k + \frac{i-|i|}{2}\right)\Delta T}\,\Gamma_p^D(f,\alpha)\right\}$$

$$= e^{-i\pi j\,\frac{\alpha-1}{\alpha}f\Delta T}\left\{\frac{1}{n}\sum_{k=1}^{n-|i|} e^{-i\pi j\,\frac{\alpha-1}{\alpha}f(n-|i|+1-2k)\Delta T}\right\}\Gamma_p^D(f,\alpha) \quad .$$

But

$$\frac{\sin\left[\pi(n-|i|)a\right]}{\sin\left[\pi a\right]} = \frac{e^{\pi j(n-|i|)a} - e^{-\pi j(n-|i|a}}{e^{\pi ja} - e^{-\pi ja}}$$

$$= \frac{b^{n-|i|} - b^{-n+|i|}}{b - b^{-1}}$$

$$= b^{n-|i|-1} + b^{n-|i|-3} + \ldots + b^{-n+|i|+3} + b^{-n+|i|+1}$$

$$= e^{\pi j(n-|i|-1)a} + e^{\pi j(n-|i|-3)a} + \ldots + e^{\pi j(-n+|i|+3)a} + e^{\pi j(-n+|i|+1)a}$$

Note that this is obviously a sum of cosines; furthermore, comparison of this identity with the above equation for $\Gamma_i^D(f,\alpha)$ yields Eq. 17 of the main text if $a = \frac{\alpha-1}{\alpha} f\Delta T$.

Now consider conditions that will ensure that the γ_i^D will overlap by only a negligible amount in the ambiguity function. Consider first $i > 0$ and $\alpha > 1$. When the spread of γ_p^D in τ is small compared with ΔT , $\gamma_{i-1}^D[\tau-(i-1)\Delta T,\alpha]$ will not overlap $\gamma_i^D(\tau-i\Delta T,\alpha)$ if

$$i-1+\frac{\alpha-1}{\alpha}\left(\frac{n+1}{2} - k\right) < i + \frac{\alpha-1}{\alpha}\left(\frac{n+1}{2} - k\right)$$

for all k. Use of the smallest k on the left side of this inequality and the largest on the right side implies that the condition for non-overlap is

$$\frac{\alpha-1}{\alpha}\left(\frac{n-1}{2}\right) - 1 < -\frac{\alpha-1}{\alpha}\left(\frac{n-1}{2}\right)$$

or

$$\frac{\alpha-1}{\alpha} < \frac{1}{n-1} .$$

On the other hand, for $i > 0$ and $\alpha < 1$, the largest k must be used on the left side and the smallest on the right side; hence the condition is

$$-\frac{\alpha-1}{\alpha}\left(\frac{n-1}{2}\right) - 1 < \frac{\alpha-1}{\alpha}\left(\frac{n-1}{2}\right)$$

or

$$-\frac{\alpha-1}{\alpha} < \frac{1}{n-1} .$$

These two conditions can be combined to yield the condition $|\alpha-1|/\alpha < 1/(n-1)$. For $i \leq 0$ the same analysis leads to the constraint $|\alpha-1|/\alpha < 1/n$. Since the latter constraint is more stringent than the former, it is the applicable constraint.

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APPENDIX C

DERIVATION OF THE APPROXIMATE MATCHED FILTER BANKS

From Eq. 9 of the main text for $x_p(t) = x_{FM}(t)$

$$\phi_{PT}(\tau,\nu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{-\infty}^{\infty} e^{-2\pi j\nu(t-\tau)}\, x_p^*[t-\tau-(i-\tfrac{n+1}{2})\Delta T]\, y(t)\, dt.$$

But

$$x_{FM}(t) \triangleq \frac{1}{\sqrt{T}} \sqcap (t;T/2)\, e^{+2\pi j(f_o t+\frac{1}{2} kt^2)} \quad ,$$

implies Eq. 22 of the main text; hence

$$\phi_{PT}(\tau,\nu) \cong \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{-\infty}^{\infty} e^{-2\pi j[\nu(i-\frac{n-1}{2})\,\Delta T\, -\, \frac{\nu f_o}{k}]}\, x_{FM}^*[t+\tfrac{\nu}{k}-\tau-(i-\tfrac{n+1}{2})\Delta T]\, y(t)\, dt \quad .$$

Substitution of the definition Eq. 24 of the main text of $\phi_{FM}(\tau,o)$ into this result yields Eq. 23 of the main text.

Also from Eq. 9 of the main text for $x_p(t) = x_{FM}(t)$

$$\phi_{PT}^{D}(\tau,\alpha) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{-\infty}^{\infty} x_p^*\,[\alpha(t-\tau) - (i-\tfrac{n+1}{2})\Delta T]\, y(t)\, dt \quad .$$

But the definition of $x_{FM}(t)$ given above implies Eq. 25 of the main text; hence

$$\phi_{PT}^{D}(\tau,\alpha) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{-\infty}^{\infty} e^{+2\pi j\, \frac{(\alpha-1)f_o^2}{k}}\, x_{FM}^*[t+(\alpha-1)f_o/k\, -\tau-(i-\tfrac{n+1}{2})\Delta T/\alpha]\, y(t)\, dt \ .$$

Substitution of the definition Eq. 24 of the main text of $\phi_{FM}(\tau,o)$ into this result yields Eq. 26 of the main text.

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